Comparison of CP-Based Single Carrier and OFDM With Power Allocation

Neng Wang, *Student Member, IEEE,* and Steven D. Blostein, *Senior Member, IEEE*

***Abstract*—Existing performance comparisons of single-carrier (SC) and multicarrier transmission are mainly based on simulations. In this letter, we present a framework to compare the performance of cyclic prefix (CP)-based SC block transmission and orthogonal frequency-division multiplexing (OFDM) in frequency-selective fading channels. Minimum mean-squared error, equal-gain, and approximate minimum bit-error rate power-allocation schemes for CP-OFDM are considered. Performance of these methods is compared analytically and verified by simulations.**

***Index Terms*—Cyclic prefix (CP), frequency-selective fading, orthogonal frequency-division multiplexing (OFDM), single-carrier (SC), power allocation.**

I. Introduction

COMPARISONS between multicarrier (MC), particularly orthogonal frequency-division multiplexing (OFDM), and single-carrier (SC) transmission has been investigated over the years. Simulations [1] have shown that the uncoded performance of SC transmission with frequency-domain equalization (FDE) substantially outperforms that of OFDM using a cyclic prefix (CP). In [1], powerful channel coding and frequency-domain interleaving are used to combat performance degradation from channel nulls in OFDM. Recently, it has been shown that block SC transmission, with similar block structure to CP-OFDM and FDE, has similar equalization complexity and coded performance to that of OFDM [2]. In this letter, we present a framework for analytical comparison between CP-SC block transmission and CP-OFDM with power allocation. Power allocation adjusts the transmitted power of subcarriers, which is a simplified alternative to joint power and bit loading widely studied in discrete multitone (DMT), an MC scheme used in wire-line applications [3]. In this letter, a variety of power-allocation schemes for CP-OFDM are analyzed and compared, including approximate minimum bit-error rate (AMBER), equal-gain (EG), and minimum mean-squared error (MMSE) [4], [5].

II. Background

We review a variety of transmission schemes. Zero-forcing (ZF) equalization is considered, since its error-rate performance can be found in closed form. For CP-OFDM, it can be shown that MMSE equalization and ZF equalization are equivalent [5]. For CP-SC using MMSE equalization, though no closed-form error-rate performance can be found, by using the Beaulieu series [6], the instantaneous bit-error rate (BER) can be approximated closely with a fast Fourier transform (FFT)-based efficient calculation [5].

A serial stream of data is serial-to-parallel (S/P) converted into data blocks of size $N, \boldsymbol{s}$. For OFDM transmission, MC modulation is performed via inverse fast Fourier transform (IFFT) to obtain the time-domain signal block, $\boldsymbol{F}^H\boldsymbol{s}$, where $\boldsymbol{F}$ is the $N \times N$ FFT matrix with (m,k)th entry $[\boldsymbol{F}]_{m,k} = N^{1/2}e^{-j2\pi mk/N}$, and $(\cdot)^H$ denotes matrix Hermitian. For SC systems, no MC modulation is performed. A length-G CP guard interval is then inserted between each block, and the resulting block is parallel-to-serial (P/S) converted and sent sequentially through the channel. A multipath propagation channel can be modeled as a finite impulse response (FIR) filter with tap vector $[h_0, h_1, \ldots, h_L]^T$ and additive white Gaussian noise (AWGN) $\eta \sim \mathcal{N}(0, N_0)$. At the receiver, the first G entries corresponding to the CP are removed. For simplicity of analysis, we assume that both input and noise are white, the length of the CP G is no less than the channel model order L, and binary phase-shift keying (BPSK) modulation is used.

A. SC With CP (CP-SC)

The received block of CP-SC can be written as (after CP removal)

$$\boldsymbol{r}_{\mathrm{SC}} = \boldsymbol{H}\boldsymbol{s} + \boldsymbol{\eta} \tag{1}$$

where $\boldsymbol{H}$ is an $N \times N$ circulant channel matrix with first column $\boldsymbol{h} \stackrel{\mathrm{def}}{=} [h_0, \ldots, h_L, 0, \ldots, 0]^T$, and $\boldsymbol{\eta}$ is the $N \times 1$ AWGN vector. By assumption, $\boldsymbol{R}_{ss} \stackrel{\mathrm{def}}{=} \mathbb{E}\{\boldsymbol{s}\boldsymbol{s}^H\} = E_s\boldsymbol{I}$ and $\boldsymbol{R}_{\eta\eta} \stackrel{\mathrm{def}}{=} \mathbb{E}\{\boldsymbol{\eta}\boldsymbol{\eta}^H\} = N_0\boldsymbol{I}$. The ZF time-domain equalizer is given by the circulant matrix $\boldsymbol{H}^\dagger$, where $(\cdot)^\dagger$ denotes the Moore–Penrose pseudoinverse. The ZF equalization output is $\hat{\boldsymbol{s}}_{\mathrm{SC}} = \boldsymbol{s} + \boldsymbol{H}^\dagger\boldsymbol{\eta}$, from which the decision-point signal-to-noise ratio (SNR) can be calculated as

$$\gamma_k^{\mathrm{SC}} = \frac{E_s}{N_0[(\boldsymbol{H}^H\boldsymbol{H})^\dagger]_{k,k}} = \frac{\gamma_s}{\frac{1}{N}\sum_{l=0}^{N-1}|H_l|^{-2}}, \quad \forall k \tag{2}$$

where $\gamma_s \stackrel{\mathrm{def}}{=} E_s/N_0$. Using (2), the average instantaneous BER is given by[1]

$$P^{\mathrm{SC}}(\boldsymbol{h}) = Q\left(\sqrt{\frac{2\gamma_s}{\frac{1}{N}\sum_{l=0}^{N-1}|H_l|^{-2}}}\right) \tag{3}$$

where $Q(x) \stackrel{\mathrm{def}}{=} (1/\sqrt{2\pi})\int_x^\infty e^{-y^2/2}\,dy$.

[1]The noise in the decision variable is generally not AWGN. This assumes AWGN gives a good approximation to the BER performance.

Paper approved by C. Tellambura, the Editor for Modulation and Signal Design of the IEEE Communications Society. Manuscript received November 21, 2003; revised October 29, 2004. This work was supported in part by the Natural Sciences and Engineering Research Council under CRD Project Grant 226876-99, and in part by grants from Bell Mobility and Samsung Electronics. This paper was presented in part at the IEEE Globecom, San Francisco, CA, 2003.

The authors are with the Department of Electrical and Computer Engineering, Queen's University, Kingston, ON K7L 3N6, Canada (e-mail: sdb@ee.queensu.ca).

Digital Object Identifier 10.1109/TCOMM.2005.843432

B. CP-OFDM

After CP removal at the CP-OFDM receiver, the received block can be written as

$$\boldsymbol{r}_{\rm MC} = \boldsymbol{H}\boldsymbol{F}^H\boldsymbol{s} + \boldsymbol{\eta}. \tag{4}$$

Performing MC demodulation via FFT, we obtain the frequency-domain received block

$$\boldsymbol{r}^f_{\rm MC} \stackrel{\rm def}{=} \boldsymbol{F}\boldsymbol{r}_{\rm MC} = \boldsymbol{F}\boldsymbol{H}\boldsymbol{F}^H\boldsymbol{s} + \boldsymbol{F}\boldsymbol{\eta} = \boldsymbol{D}\boldsymbol{s} + \boldsymbol{\eta}^f \tag{5}$$

where the superscript ^{f} denotes a frequency-domain variable, and the last equality follows from the well-known property of circulant matrices, $\boldsymbol{F}\boldsymbol{H}\boldsymbol{F}^H = {\rm diag}(\boldsymbol{F}\boldsymbol{h}) \stackrel{\rm def}{=} \boldsymbol{D}$. The frequency-domain ZF equalizer is given by the diagonal matrix $\boldsymbol{D}^\dagger$. The average instantaneous BER is given by

$$P^{\rm MC}(\boldsymbol{h}) = \frac{1}{N}\sum_{k=0}^{N-1} Q\left(\sqrt{2\gamma_s|H_k|^2}\right). \tag{6}$$

C. CP-OFDM With Power Allocation

Denote p_k^2 as the transmitted power of the kth subcarrier ($k = 0, 1, \ldots, N-1$), and define the power-allocation matrix $\boldsymbol{P} = {\rm diag}([p_0, p_1, \ldots, p_{N-1}]^T)$. The block power constraint can be normalized as

$${\rm trace}\{\boldsymbol{P}^2\} = \sum_{k=0}^{N-1} p_k^2 = N. \tag{7}$$

From (5), the received frequency-domain block is given by $\boldsymbol{r}^f_{\rm PMC} = \boldsymbol{D}\boldsymbol{P}\boldsymbol{s} + \boldsymbol{\eta}^f$. With ZF equalization, we obtain $\hat{\boldsymbol{s}}_{\rm PMC} = \boldsymbol{s} + (\boldsymbol{D}\boldsymbol{P})^\dagger\boldsymbol{\eta}^f$. The decision-point SNR of the kth subcarrier is given by $\gamma_s|H_k|^2p_k^2$, and the average instantaneous BER

$$P(\boldsymbol{h}, \boldsymbol{P}) = \frac{1}{N}\sum_{k=0}^{N-1} Q\left(\sqrt{2\gamma_s|H_k|^2p_k^2}\right). \tag{8}$$

1) AMBER Power Allocation: To minimize (8) under transmit power constraint (7), no closed-form solution exists. However, taking the approach in [5], we approximate the objective function to obtain a closed-form solution[2]

$$p^2_{{\rm AMBER},k} = \left(\frac{\ln|H_k|^2 + \nu}{\gamma_s|H_k|^2}\right)_+ \tag{9}$$

where $(x)_+ \stackrel{\rm def}{=} \max\{0, x\}$, and ν is chosen to satisfy the block power constraint (7).

2) EG Power Allocation: The EG power-allocation solution pre-equalizes the gains of subcarriers so that all gains are equal,[3] which is given by

$$p^2_{{\rm EG},k} = \frac{|H_k|^{-2}}{\frac{1}{N}\sum_{l=0}^{N-1}|H_l|^{-2}}, \quad (k = 0, \ldots, N-1). \tag{10}$$

3) MMSE Power Allocation: The MMSE power allocation minimizes the MSE of the ZF equalizer output. Using Lagrange multipliers, the MMSE power-allocation solution under the block power constraint (7) can be found as [5]

$$p^2_{{\rm MMSE},k} = \frac{|H_k|^{-1}}{\frac{1}{N}\sum_{l=0}^{N-1}|H_l|^{-1}}, \quad (k = 0, \ldots, N-1). \tag{11}$$

III. Performance Comparison

We compare the instantaneous BER of ZF-equalized CP-SC and a variety of CP-OFDM schemes.

A. CP-SC Versus CP-OFDM Without Power Allocation

From (3) and (6), the average instantaneous BER expressions can be unified as

$$P\left(\{\xi_k\}_{k=0}^{N-1}\right) = \frac{1}{N}\sum_{k=0}^{N-1} Q\left(\sqrt{\frac{2\gamma_s}{\xi_k}}\right) \tag{12}$$

where ξ_k's for CP-OFDM and CP-SC are given by, respectively

$$\xi_k^{\rm MC} = |H_k|^{-2}, \quad \xi_k^{\rm SC} = \frac{1}{N}\sum_{l=0}^{N-1}|H_l|^{-2} \stackrel{\rm def}{=} \xi^{\rm SC}. \tag{13}$$

From the second derivatives of (12), $(d^2P)/(d\xi_k^2) = (2N)^{-1}\sqrt{(\gamma_s)/(\pi\xi_k^5)}(\gamma_s/\xi_k - 3/2)e^{-\gamma_s/\xi_k}$, it can be shown that the Hessian matrix is positive definite, as long as the decision-point SNR $\gamma_k = \gamma_s/\xi_k > 3/2$ (1.76 dB). This condition holds asymptotically when γ_s is large. Therefore, $P(\{\xi_k\}_{k=0}^{N-1})$ is asymptotically convex in ξ_k's. From (13), we know $\xi^{\rm SC} = (1/N)\sum_{k=0}^{N-1}\xi_k^{\rm MC}$. By Jensen's inequality, we conclude that

$$P^{\rm SC}(\boldsymbol{h}) \le P^{\rm MC}(\boldsymbol{h}). \tag{14}$$

B. CP-SC Versus CP-OFDM With EG Power Allocation

For CP-OFDM with EG power allocation, we calculate the decision-point SNR [see (10)]

$$\gamma^{\rm PMC}_{{\rm EG},k} = \gamma_s|H_k|^2p^2_{{\rm EG},k} = \frac{\gamma_s}{\frac{1}{N}\sum_{l=0}^{N-1}|H_l|^{-2}} \equiv \gamma_k^{\rm SC}. \tag{15}$$

Therefore, CP-OFDM with EG power allocation has the same performance as ZF-equalized CP-SC.

C. CP-SC Versus CP-OFDM With MMSE Power Allocation

From (8), (11), and (12), we have $\xi^{\rm PMC}_{{\rm MMSE},k} = |H_k|^{-1}N^{-1}\sum_{l=0}^{N-1}|H_l|^{-1}$. When $\gamma_s \gg 1$, by Taylor series expansion of $P(\{\xi^{\rm PMC}_{{\rm MMSE},k}\})$ at $\xi^{\rm SC}$, we can write $P^{\rm PMC}_{\rm MMSE}(\boldsymbol{h})$ as

$$\begin{aligned} P^{\rm PMC}_{\rm MMSE}(\boldsymbol{h}) = P(\xi^{\rm SC}) &+ \left(\frac{1}{N}\sum_{k=0}^{N-1}\xi^{\rm PMC}_{{\rm MMSE},k} - \xi^{\rm SC}\right) \\ &\times \frac{dP(\xi^{\rm SC})}{d\xi} + \frac{1}{2N}\sum_{k=0}^{N-1}\left(\xi^{\rm PMC}_{{\rm MMSE},k} - \xi^{\rm SC}\right)^2 \\ &\times \frac{d^2P(\xi^{\rm SC})}{d\xi^2} + \cdots. \end{aligned} \tag{16}$$

We note that in the above series expansion, $(1/N)\sum_{k=0}^{N-1}\xi^{\rm PMC}_{{\rm MMSE},k} - \xi^{\rm SC} = (N^{-1}\sum_{l=0}^{N-1}|H_l|^{-1})^2 -$

[2]An equivalent solution was proposed in the independent work [4], where the Chernoff upper bound of BER was used.

[3]This scheme was also addressed in [3] as a high-SNR approximation to power allocation.

$N^{-1}\sum_{l=0}^{N-1}|H_l|^{-2}$, while $\sum_{k=0}^{N-1}(\xi_{\mathrm{MMSE},k}^{\mathrm{PMC}} - \xi^{\mathrm{SC}})^2 \geq 0$. However, from the first and second derivatives of $P(\xi)$, we obtain $(d^2P/d\xi^2)/(dP/d\xi) = \xi^{-1}(\gamma_s/\xi - 3/2)$. When $\gamma_s \gg 1$, we have $d^2P/d\xi^2 \gg dP/d\xi$, and, accordingly, from (16)

$$P^{\mathrm{SC}}(\boldsymbol{h}) \leq P_{\mathrm{MMSE}}^{\mathrm{PMC}}(\boldsymbol{h}). \quad (17)$$

D. CP-SC Versus CP-OFDM With AMBER Power Allocation

Note that in (9), at high SNR, the condition $\nu \geq \max_k\{-\ln|H_k|^2\}$ holds, and the AMBER power allocation can be obtained as

$$p_k^2 = \frac{\ln|H_k|^2 + \nu}{\gamma_s|H_k|^2} \quad (18a)$$

$$\nu \stackrel{\mathrm{def}}{=} \frac{\gamma_s - \frac{1}{N}\sum_{l=0}^{N-1}|H_l|^{-2}\ln|H_l|^2}{\frac{1}{N}\sum_{l=0}^{N-1}|H_l|^{-2}}. \quad (18b)$$

From (8), (12), and (18), we obtain, for AMBER power-allocated CP-OFDM

$$\xi_{\mathrm{AMBER},k}^{\mathrm{PMC}} = \frac{\frac{1}{N}\sum_{l=0}^{N-1}|H_l|^{-2}}{1 + \frac{1}{\gamma_s}\cdot\frac{1}{N}\sum_{l=0}^{N-1}|H_l|^{-2}(\ln|H_k|^2 - \ln|H_l|^2)}. \quad (19)$$

At moderate-to-high SNR, $\gamma_s \gg 1$, and we can approximate

$$\xi_{\mathrm{AMBER},k}^{\mathrm{PMC}} \approx \xi^{\mathrm{SC}}\left(1 - \frac{1}{\gamma_s}\cdot\frac{1}{N}\sum_{l=0}^{N-1}\frac{\ln|H_k|^2 - \ln|H_l|^2}{|H_l|^2}\right). \quad (20)$$

From the Taylor series expansion of $P_{\mathrm{AMBER}}^{\mathrm{PMC}}(\boldsymbol{h})$ at ξ^{SC}, we have

$$\begin{aligned} P_{\mathrm{AMBER}}^{\mathrm{PMC}}(\boldsymbol{h}) &\approx P(\xi^{\mathrm{SC}}) - \frac{\xi^{\mathrm{SC}}}{\gamma_s}\frac{dP\left(\xi_{\mathrm{ZF}}^{\mathrm{SC}}\right)}{d\xi} \\ &\quad \times \underbrace{\left(\frac{1}{N}\sum_{l=0}^{N-1}\ln|H_l|^2\cdot\frac{1}{N}\sum_{l=0}^{N-1}\frac{1}{|H_l|^2} - \frac{1}{N}\sum_{l=0}^{N-1}\frac{\ln|H_l|^2}{|H_l|^2}\right)}_{\geq 0} \\ &\leq P(\xi^{\mathrm{SC}}) = P^{\mathrm{SC}}(\boldsymbol{h}). \end{aligned}$$

IV. Numerical Results

We now compare our analytical asymptotic performance results derived in Section III with simulations at finite SNRs.

Example 1: We compare the instantaneous BER performance of SC and a variety of OFDM schemes. The frequency-selective channel tap vector is randomly generated. The instantaneous BER performance is plotted in Fig. 1. At all SNRs shown, CP-OFDM with AMBER power allocation outperforms ZF-equalized CP-SC (or CP-OFDM with EG power allocation), CP-OFDM with MMSE power allocation, and conventional CP-OFDM without power allocation.

Example 2: We compare the BER performance of SC and MC schemes in fading channels. The channel models in [7] are used. Channel 1 with tap powers [0.15, 0.65, 0.15, 0.05] has moderate nulls, while channel 2 with tap powers [0.39, 0.16, 0.26, 0.19] has severe nulls [7]. From Figs. 2 and 3, we see that CP-OFDM with AMBER power allocation outperforms

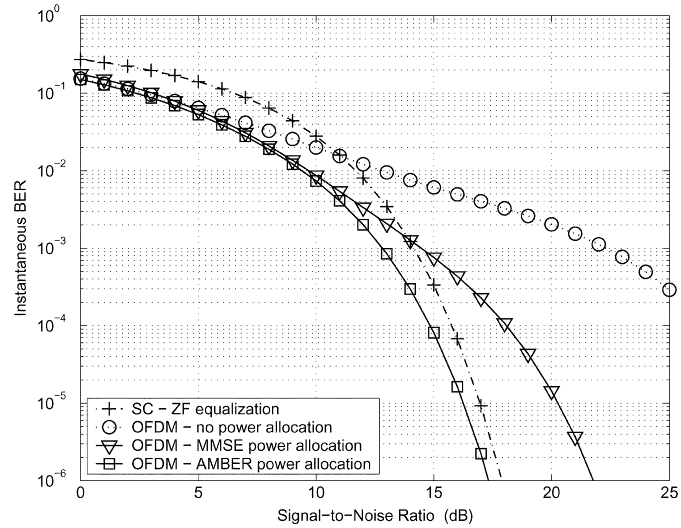

Fig. 1. Example of instantaneous BER comparison of SC and MC transmission in a frequency-selective channel (block size $N = 64$).

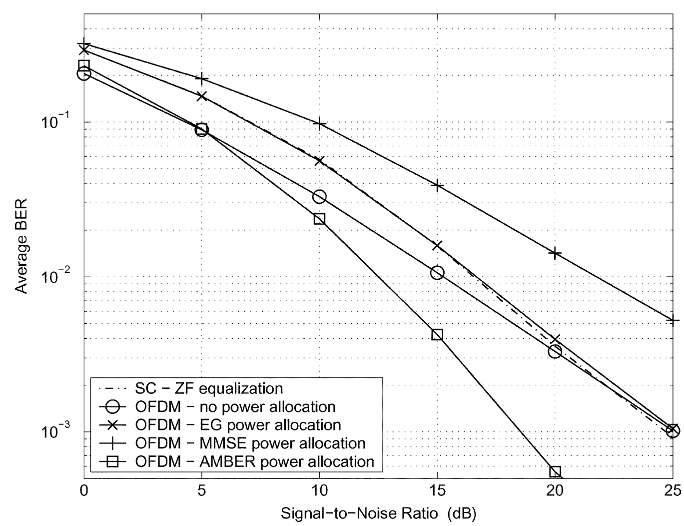

Fig. 2. BER comparison using frequency-selective fading channel model 1 in [7] (block size $N = 16$).

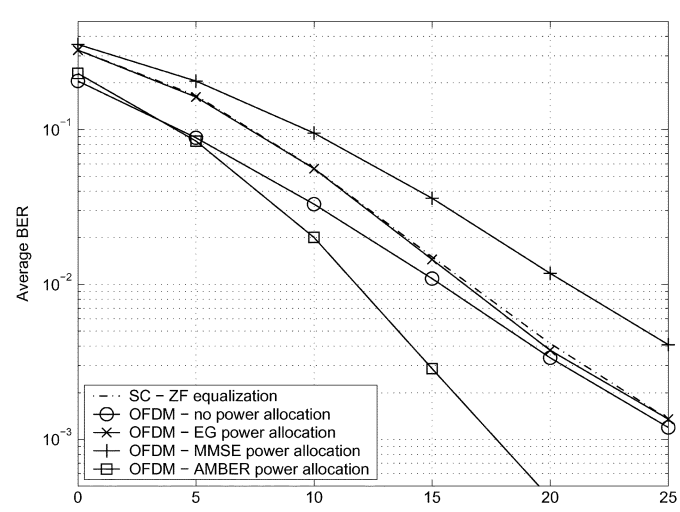

Fig. 3. BER comparison using frequency-selective fading channel model 2 in [7] (block size $N = 16$).

ZF-equalized SC (or OFDM with EG power allocation) and conventional CP-OFDM without power allocation. At a BER of 10^{-3}, CP-OFDM with AMBER power allocation has a gain of around 7.5 dB as compared with EG power allocation, as well as schemes without power allocation. We also observe from these figures that CP-OFDM with MMSE power allocation has poor BER performance in fading channels at all SNRs shown.

V. Conclusion

CP-based SC and OFDM transmission schemes are compared in this letter. Through analysis and confirmed by simulations, we have established that uncoded CP-OFDM is inferior to ZF-equalized CP-SC, CP-OFDM with EG power allocation has the same performance as ZF-equalized CP-SC, and CP-OFDM with AMBER power allocation outperforms ZF-equalized CP-SC. It is worth noting that the complexities of the different power-allocation schemes are comparable.

References

[1] H. Sari, G. Karam, and I. Jeanclaude, "Frequency-domain equalization of mobile radio and terrestrial broadcast channels," in *Proc. IEEE Globecom*, vol. 1, San Francisco, CA, Nov.–Dec. 1994, pp. 1–5.

[2] D. Falconer, S. L. Ariyavisitakul, A. Benyamin-Seeyar, and B. Eidson, "Frequency domain equalization for single-carrier broadband wireless systems," *IEEE Commun. Mag.*, vol. 40, no. 4, pp. 58–66, Apr. 2002.

[3] J. A. C. Bingham, "Multicarrier modulation for data transmission: An idea whose time has come," *IEEE Commun. Mag.*, vol. 28, no. 5, pp. 6–14, May 1990.

[4] A. P. Iserte, M. A. Lagunas, and A. I. Pérez-Neira, "Robust power allocation for minimum BER in a SISO-OFDM system," in *Proc. IEEE ICC*, vol. 2, Anchorage, AK, May 2003, pp. 1263–1267.

[5] N. Wang and S. D. Blostein, "Power loading for CP-OFDM over frequency-selective fading channels," in *Proc. IEEE Globecom*, vol. 4, San Francisco, CA, Dec. 2003, pp. 2305–2309.

[6] N. C. Beaulieu, "The evaluation of error probabilities for intersymbol and cochannel interference," *IEEE Trans. Commun.*, vol. 39, no. 12, pp. 1740–1749, Dec. 1991.

[7] D. Kim and G. L. Stüber, "Residual ISI cancellation for OFDM with application to HDTV broadcasting," *IEEE J. Sel. Areas Commun.*, vol. 16, no. 8, pp. 1590–1599, Oct. 1998.