Density and Bounds for Grassmannian Codes with Chordal Distance

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***Abstract*—We investigate the density of codes in the complex Grassmann manifolds $G^{\mathbb{C}}_{n,p}$ equipped with the chordal distance. The density of a code is defined as the fraction of the Grassmannian covered by 'kissing' balls of equal radius centered around the codewords. The kissing radius cannot be determined solely from the minimum distance, nonetheless upper and lower bounds as a function of minimum distance only are provided, along with the corresponding bounds on the density. This leads to a refinement of the Hamming bound for Grassmannian codes. Finally, we provide explicit bounds on code cardinality and minimum distance, notably a generalization of a bound on minimum distance previously proven only for line packing ($p = 1$).**

I. INTRODUCTION

Grassmannian codes are a generalization of spherical codes with applications in the area of multiple-antenna transmission [1]–[4]. Results on the structure of the Grassmann manifold have application to coherent MIMO systems with limited feedback, non-coherent MIMO systems, as well as manifold signal processing. Density is an important property for spherical codes. In [3], using chordal distance as a metric, the density of a line packing is defined. However, when considering packing of complex lines in $\mathbb{C}^2$, the complex projective line is ismorphic to the Riemann sphere [5, Ex.17.23], and the proposed definition is not consistent with the conventional definition of density for spherical codes [6]. This comes from the fact that the chordal distance is not "strictly intrinsic," i.e., the triangle inequality is never satisfied with equality, as remarked in [7].

In this paper, we investigate a density of Grassmannian codes which is consistent with the density of spherical codes. Except in the case of line packing where a closed-form solution is provided, the density is not a single-variable function of the minimum distance of the code. Namely, two codes with equal minimum distance could have different densities. Nevertheless, upper and lower bounds on the density as a function of minimum distance only are provided. In order to obtain these bounds on the code density, we bound the *kissing radius*, i.e., the maximum radius of non-overlapping metric balls centered around the codewords (so called *sphere packing*).

Another fundamental problem of coding theory is to establish the relationship between the size of the code and the minimum distance among elements of the code. The Hamming bound gives an upper bound of the size of codes for a given minimum distance. In previous work, asymptotic results on the volume of a metric ball in the Grassmann manifold $G^{\mathbb{C}}_{n,p}$ as $n \to \infty$ were derived [7] leading to bounds on the asymptotic rate of codes. These bounds were later improved in [8], [9]. The framework in [7] was generalized and bounds were derived for the minimum distance on codes for arbitrary values of n and p in [10].

The approach to investigating density presented here leads to a refinement of the Hamming bound for any values of n and p. The Hamming bound provided in this paper is tighter than the Hamming bound that can be derived using the geometrical argument in [7].

Finally, we use the Hamming bound to compute two explicit bounds on codes in complex Grassmann manifolds by exploiting the closed-form expression on the volume of a small ball derived in [11]. We first derive an explicit bound on code cardinality given the minimum distance. Then, we provide a new upper bound on the minimum distance tighter than the Rankin bound for large codebook sizes. This bound appears to be a generalization of a bound on minimum distance previously proven only for line packing ($p = 1$) in [12, Eq. (32)].

II. DEFINITION AND EARLIER RESULTS

A. Grassmann manifold

The complex Grassmann manifold $G^{\mathbb{C}}_{n,p}$, with $p \leq n$, is the set of p-dimensional subspaces in the n-dimensional complex vector space $\mathbb{C}^n$. It can be expressed as a homogeneous space of the unitary group $\mathcal{U}_p$:

$$G^{\mathbb{C}}_{n,p} \cong \frac{V^{\mathbb{C}}_{n,p}}{\mathcal{U}_p} \tag{1}$$

where $V^{\mathbb{C}}_{n,p}$ is the complex Stiefel manifold, the space of orthonormal non-square matrices:

$$V^{\mathbb{C}}_{n,p} = \left\{ Y \in \mathbb{C}^{n \times p} \quad | \quad Y^H Y = I_p \right\}.$$

The quotient representation in (1) implies that a point in the Grassmann manifold can be represented by the equivalence class of $n \times p$ unitary matrices whose columns span the same space:

$$[Y] = \{ Y U_p \quad | \quad U_p \in \mathcal{U}_p \}$$

with $Y \in V^{\mathbb{C}}_{n,p}$, a generator of the Grassmannian plane $[Y]$. Because $G^{\mathbb{C}}_{n,p}$ and $G^{\mathbb{C}}_{n,n-p}$ are isomorphic, we will assume that $p \leq n/2$.

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Given Y, $Z \in V_{n,p}$, we denote the principal angles [13] between the two subspaces of $\mathrm{G}^{\mathbb{C}}_{n,p}$, $[Y]$, $[Z] \in \mathrm{G}_{n,p}$ by $\theta_1 \ldots \theta_p \in [0, \frac{\pi}{2}]$. To the principal angles, we associate the p-dimensional vectors $\vec{\theta} = [\theta_1, \cdots, \theta_p]^T$, $\sin\vec{\theta} = [\sin\theta_1, \cdots, \sin\theta_p]^T$ and $\cos\vec{\theta} = [\cos\theta_1, \cdots, \cos\theta_p]^T$. The principal angles are independent of the possible generators and may be calculated from the the singular value decomposition of the matrix $YZ^\dagger$ which has the form $U\Sigma V^\dagger$ where $\Sigma = \mathrm{diag}(\cos\vec{\theta})$, where $M^\dagger$ denotes the Hermitian conjugate of M.

Several nonequivalent distances can be defined on the Grassmann manifold [14]. In this paper, we are primary interested in *chordal distance*. The chordal distance arises naturally by embedding the Grassmann manifold into the space of associated projection matrices [15]. For each $[Y] \in \mathrm{G}_{n,p}$, we associate the orthogonal projection from $\mathbb{C}^n$ to $[Y]$: $\Pi_Y = YY^\dagger$. This projection is unique for every element of $\mathrm{G}^{\mathbb{C}}_{n,p}$ and independent of the generator matrix. Projection matrices are $n \times n$ idempotent matrices which lie in the space of n^2-1-dimensional Hermitian matrices with trace equal to p [10]. The following map is an isometric embedding [15]

$$\begin{array}{rcl} \mathrm{G}^{\mathbb{C}}_{n,p} & \hookrightarrow & S^{n^2-2}\left(\sqrt{\frac{p(n-p)}{2n}}\right) \subset \mathbb{R}^{n^2-1} \\ {[Y]} & \mapsto & \frac{1}{\sqrt{2}}(\Pi_Y - \frac{p}{n}I) \end{array} \tag{2}$$

where the Grassmannian $\mathrm{G}^{\mathbb{C}}_{n,p}$ is equipped with the following *chordal distance* metric:

$$d_c([Y],[Z]) = \frac{1}{\sqrt{2}}||\Pi_Y - \Pi_Z||_F. \tag{3}$$

The chordal distance between $[Y]$ and $[Z] \in \mathrm{G}^{\mathbb{C}}_{n,p}$ can be also expressed as function of the principal angles [15]

$$d_c([Y],[Z]) = \|\sin\vec{\theta}\|_2. \tag{4}$$

We denote $B_{[Y]}(\gamma)$ as the metric ball of radius γ with center at $[Y]$, defined as

$$B_{[Y]}(\gamma) = \left\{[V] \in \mathrm{G}^{\mathbb{C}}_{n,p} \ : \ d_c([V],[Y]) \leq \gamma\right\}. \tag{5}$$

The normalized volume of a ball of radius $\gamma \leq 1$ has been derived in closed form in [11], generalizing the result of [4]:

$$\mu(B(\gamma)) = c_{n,p}\gamma^{2p(n-p)} \tag{6}$$

where

$$c_{n,p} = \frac{1}{p(n-p)!}\prod_{i=1}^{p}\frac{(n-i)!}{(p-i)!}. \tag{7}$$

This quantity is independent of its center $[Y]$.

B. Grassmannian codes

An (N,δ) code in $\mathrm{G}^{\mathbb{C}}_{n,p}$ is a finite subset of N points in the Grassmannian with (pairwise) minimum distance among the elements δ. A packing is a code that maximizes the minimum distance for a given cardinality.

Since a Grassmannian code can be isometrically embedded in a hypersphere, the Rankin bound [16] on spherical codes leads to the following bound on the minimum distance of complex Grassmannian codes [15]

$$\delta^2 \leq \begin{cases} \frac{p(n-p)}{n}\frac{N}{N-1} & \text{if } N \leq n^2 \\ \frac{p(n-p)}{n} & \text{if } n^2 < N \leq 2(n^2-1) \end{cases}. \tag{8}$$

The above bound is still valid but not achievable for $N > 2(n^2-1)$, motivating the search for lower bounds for higher value of N.

For any (N,δ) code in $\mathrm{G}^{\mathbb{C}}_{n,p}$, the standard Hamming bound states that

$$N\mu(B(\delta/2)) \leq 1. \tag{9}$$

Applying the volume formula (6), an upper bound of the size of codes for a given minimum distance can be expressed as follows [11]: when $\delta \leq 2$, for any (N,δ) complex Grassmannian code

$$N \leq c_{n,p}^{-1}\left(\frac{\delta}{2}\right)^{-2p(n-p)}. \tag{10}$$

As noted in [7], it is possible to extend the radius $\frac{\delta}{2}$ so that the Hamming bound is still valid.

Lemma 1 ([7]): For any (N,δ) code in $\mathrm{G}^{\mathbb{C}}_{n,p}$, define $r = \sqrt{\frac{p(n-p)}{2n}}$. Then

$$N\mu(B(\delta_m)) \leq 1 \tag{11}$$

with

$$\delta_m = \sqrt{2}r\sqrt{1-\sqrt{1-\frac{\delta^2}{4r^2}}}. \tag{12}$$

Proof: From the isometric embedding, given $[Y]$, $[Z] \in \mathrm{G}^{\mathbb{C}}_{n,p}$ with distance δ, simple geometric arguments show that the midpoint between their corresponding images on $S^{n^2-2}(r)$ is at distance δ_m given in (12) [7]. The balls $B_{[Y]}(\delta_m)$ and $B_{[Z]}(\delta_m)$ have thus no common interior. ∎

The volume of the ball of radius δ_m can then be evaluated in closed form with the help of (6) provided that $\delta_m \leq 1$, which is satisfied if either a) $2r^2 \leq 1 \Leftrightarrow p = 1$, or $(n,p) = (4,2)$ or b) $\delta^2 \leq 4 - r^{-2}$.

Corollary 1: For any (N,δ) code in $\mathrm{G}^{\mathbb{C}}_{n,p}$ with $p = 1$, $(n,p) = (4,2)$, or $\delta \leq \sqrt{4-r^{-2}}$ with $r^2 = \frac{p(n-p)}{2n}$,

$$N \leq c_{n,p}^{-1}\left[2r^2\left(1-\sqrt{1-\frac{\delta^2}{4r^2}}\right)\right]^{-p(n-p)}. \tag{13}$$

III. Density of Grassmannian code

We define the density of a Grassmannian code as the fraction of $\mathrm{G}^{\mathbb{C}}_{n,p}$ covered by non-overlapping metric balls of maximum equal radius centered around the codewords.

Definition 1: The density of a code $[\mathcal{W}] \in \mathrm{G}^{\mathbb{C}}_{n,p}$ is defined as

$$\Delta([\mathcal{W}]) = N\mu\left(B(\varrho_{[\mathcal{W}]})\right) \tag{14}$$

with

$$\varrho_{[\mathcal{W}]} = \sup_{\substack{B_{[W_l]}(\gamma)\cap B_{[W_k]}(\gamma)=\emptyset \\ \forall(k,l)\ k\neq l}} \gamma. \tag{15}$$

By definition, we have $\Delta([\mathcal{W}]) \leq 1$. The kissing radius $\varrho_{[\mathcal{W}]}$ is hard to determine since it not only depends on the minimum distance of the packing but on the principal angles between the subspaces. However, the next result provides upper and lower bounds on $\varrho_{[\mathcal{W}]}$ and corresponding bounds on the density.

Proposition 1: For any (N, δ) code in $G^{\mathbb{C}}_{n,p}$, we have

$$\underline{\varrho} \leq \varrho_{[\mathcal{W}]} \leq \bar{\varrho} \tag{16}$$

where

$$\underline{\varrho} = \sqrt{\frac{p}{2}\left(1 - \sqrt{1 - \frac{\delta^2}{p}}\right)}, \tag{17}$$

$$\bar{\varrho} = \sqrt{\frac{1}{2}\left(\lceil \delta^2 \rceil - \sqrt{\lceil \delta^2 \rceil - \delta^2}\right)}, \tag{18}$$

and $\lceil x \rceil$ is the smallest integer greater or equal to x. It follows, therefore, that

$$N\mu\left(B(\underline{\varrho})\right) \leq \Delta([\mathcal{W}]) \leq \min\left\{1, N\mu\left(B(\bar{\varrho})\right)\right\}. \tag{19}$$

A proof is provided in appendix.

Figure 1 illustrates the upper and lower bounds (16) on the kissing radius and shows that $\delta/2$ is a good approximation for $\varrho_{[\mathcal{W}]}$ when the minimum distance of the code is relatively small. In general, since the chordal distance is not strictly intrinsic, we have $\frac{\delta}{2} < \varrho_{[\mathcal{W}]}$.

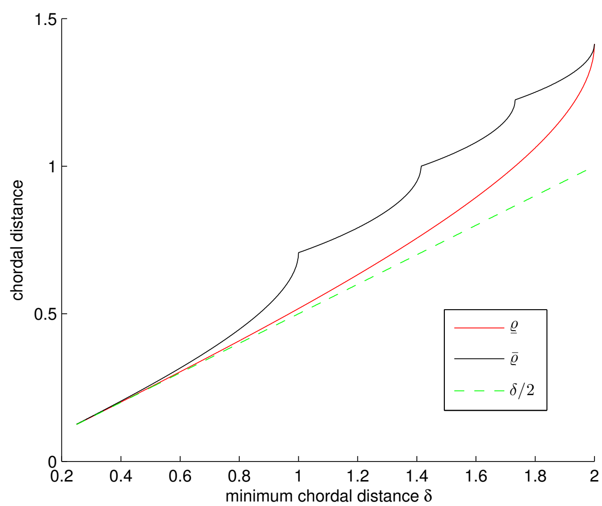

Fig. 1. From top to bottom, upper and lower bounds (16), and approximation $\delta/2$ of the kissing radius $\varrho_{[\mathcal{W}]}$ for $p = 4$.

Bounds (19) on the density may be calculated in a more explicit form using of (6), providing that $\bar{\varrho} \leq 1$ and $\underline{\varrho} \leq 1$.

Corollary 2: For any (N, δ) code in $G^{\mathbb{C}}_{n,p}$ that satisfies $\delta \leq 2\sqrt{\frac{p-1}{p}}$ or if $p \leq 2$

$$\Delta([\mathcal{W}]) \geq N c_{n,p} \underline{\varrho}^{2p(n-p)}. \tag{20}$$

Moreover for $\delta \leq 2$,

$$\Delta([\mathcal{W}]) \leq \min\left\{1,\ N c_{n,p} \bar{\varrho}^{2p(n-p)}\right\}. \tag{21}$$

In the specific case of Grassmannian line packing, i.e., $p = 1$, the kissing radius and the density can be calculated exactly as a function of the minimum distance

Corollary 3: For any line packing in $G^{\mathbb{C}}_{n,1}$

$$\varrho_{[\mathcal{W}]} = \underline{\varrho} = \bar{\varrho} \tag{22}$$

and

$$\Delta([\mathcal{W}]) = N\left(\frac{1 - \sqrt{1 - \delta^2}}{2}\right)^{n-1}. \tag{23}$$

Remark 1: For the specific case of $n = 2, p = 1$, the Grassmann manifold is isomorphic to the real sphere: $G^{\mathbb{C}}_{2,1} \cong S^2$ [5, Ex. 17.23], and the present definition of density of Grassmannian packing is consistent with the definition of density for sphere packing [6]. For illustration, in Figure 2 we have plotted the densities of the best known packings on S^2 from the literature [6].

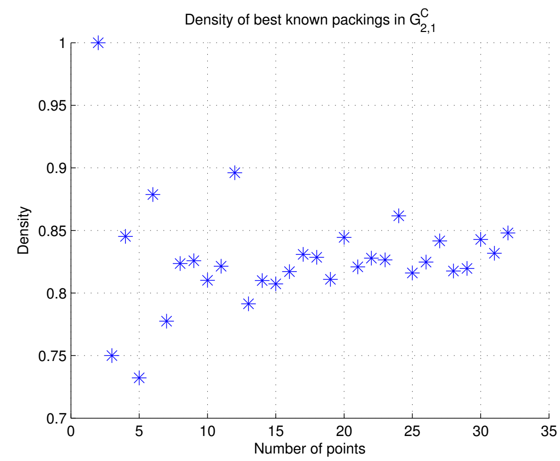

Fig. 2. Density of best known packings on $G^{\mathbb{C}}_{2,1} \cong S^2$ as a function of the number of points.

IV. Explicit Bounds on Grassmannian Codes

In this section, we exploit results on density to derive explicit bounds on cardinality and minimum distance. In the previous section, we have found a function of the minimum distance such that $f(\delta) \geq \delta/2$ still satisfies the Hamming bound $N\mu(B(f(\delta))) \leq 1$. For this we have defined the density of codes which verifies $\Delta \leq 1$, in order to bound the density from below by $N\mu(B(f(\delta))) \leq \Delta$. Then, provided that $f(\delta) < 1$, we apply the volume formula (6) to derive a closed-form upper bound on the size of the code as a function of the minimum distance $N \leq c^{-1}_{n,p}\left(f(\delta)\right)^{-2p(n-p)}$. Additionally, if f is invertible, it is possible to bound the minimum distance from above by $\delta \leq f^{-1}\left((N c_{n,p})^{\frac{-1}{2p(n-p)}}\right)$.

According to Proposition 1, we have the following improvement of the Hamming bound:

Corollary 4: For any (N, δ) code in $G^{\mathbb{C}}_{n,p}$

$$N\mu(B(\underline{\varrho})) \leq 1 \tag{24}$$

with

$$\underline{\varrho} = \sqrt{\frac{p}{2}\left(1 - \sqrt{1 - \frac{\delta^2}{p}}\right)} \qquad (25)$$

Corollary 5: The bound (24) is tighter than the bound (11) with equality if and only if $p = n/2$.

Proof: It is easy to verify that $p \leq 4r^2$ with equality iff $p = n/2$. Then, since $x/2(1 - \sqrt{1 - \delta^2/x})$ is a strictly decreasing function, it follows that $\delta_m \leq \underline{\varrho}$ with equality iff $p = n/2$. Here δ_m and r are defined in Lemma 1. ■

A. Cardinality

Accordingly, we have the following improvement of the explicit Hamming-type bound on the cardinality:

Lemma 2: For any (N, δ) code in $G^{\mathbb{C}}_{n,p}$ with $p \leq 2$, or $\delta \leq 2\sqrt{\frac{p-1}{p}}$

$$N \leq c_{n,p}^{-1}\left[\frac{p}{2}\left(1 - \sqrt{1 - \frac{\delta^2}{p}}\right)\right]^{-p(n-p)}. \qquad (26)$$

Proof: From (19), we have $N\mu\left(B(\underline{\varrho})\right) \leq 1$. The volume of the ball of radius $\underline{\varrho}$ can be evaluated in closed form with the help of (6) provided that $\underline{\varrho} \leq 1$, which is satisfied if either a) $p \leq 2$ or b) $\delta \leq 2\sqrt{\frac{p-1}{p}}$. The result then follows. ■

In Figure 3, the bound (26) is compared with bounds (10) and (13) for the Grassmannian $G^{\mathbb{C}}_{5,2}$. As can be noted, bound (26) provides a refinement for larger values of minimal distance. When the minimum distance tends to zero all the bounds converge.

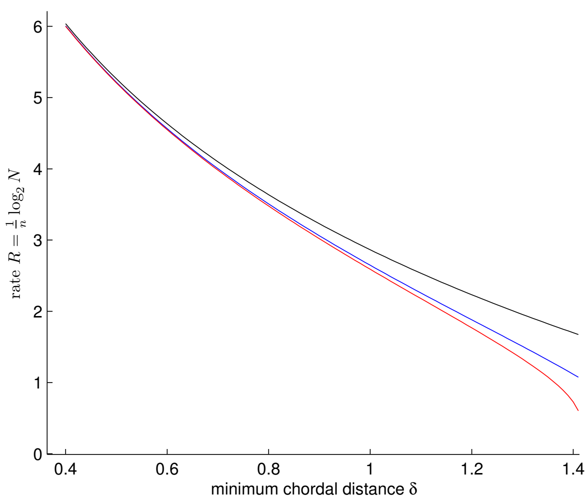

Fig. 3. Hamming bounds on the rate of codes in $G^{\mathbb{C}}_{5,2}$. From top to bottom, bounds (10), (13), and the new bound (26).

B. Minimum distance

The last result is a new bound on the minimum distance. This bounds appears to be a generalization of a bound for line packing in [12] to any value of p.

Lemma 3: For N sufficiently large, specifically for $p \leq 2$ with $N \geq c_{n,p}^{-1}\left(\frac{2}{p}\right)^{p(n-p)}$,

$$\delta^2 \leq 4(Nc_{n,p})^{\frac{-1}{p(n-p)}} - \frac{4}{p}(Nc_{n,p})^{\frac{-2}{p(n-p)}}. \qquad (27)$$

Proof: Bound (20) requires that $p \leq 2$ or $\delta \leq 2\sqrt{\frac{p-1}{p}}$. Further simple manipulations result in $N \geq c_{n,p}^{-1}\left(\frac{2}{p}\right)^{p(n-p)}$. Condition $\delta \leq 2\sqrt{\frac{p-1}{p}}$ is achievable for any code as $N \to \infty$. ■

Remark 2: For the case $p = 1$ the bound of Lemma 3 reduces to the following bound derived in [12, (32)]:

$$\delta^2 \leq 4N^{\frac{-1}{n-1}} - 4N^{\frac{-2}{n-1}}. \qquad (28)$$

As illustrated in Figure 4, this bound is tighter than the Rankin bound for very large codebook sizes.

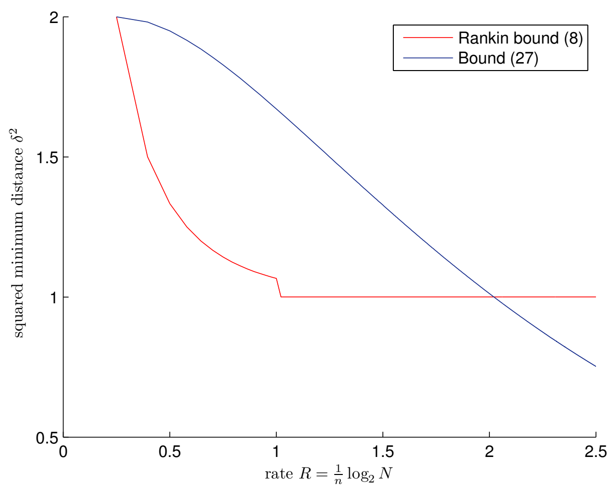

Fig. 4. Bound (27) on the minimum distance for $G^{\mathbb{C}}_{4,2}$.

V. Conclusion

This paper discussed the density and the 'kissing radius' of Grassmannian codes with chordal distance. For a code with given minimum distance and cardinality, the density is a function of the kissing radius which itself does not only depend on the minimum distance but also on the principal angles. According to this observation, we provide upper and lower bounds (as a function of the minimum distance only) on density and kissing radius for packing of equal spheres in a complex Grassmann manifold $G^{\mathbb{C}}_{n,p}$. The proof is based on a geometric argument leading to a simple optimization problem. As a direct consequence, this result gives tighter Hamming-type bounds for finite values of n, p than previously known results, i.e. in the pre-asymptotic regime. In the asymptotic regime, the analysis provided here would lead to the same bound on code rate than in [8] improving [7], but later improved in [9].

APPENDIX: PROOF OF PROPOSITION 1

Let us consider a Grassmannian code $[\mathcal{W}]$ with minimum distance δ. The principal angles $(\theta_1, \ldots, \theta_p)$ between two codewords $[W_i]$, $[W_j]$ separated by δ satisfies $\sum_{i=1}^{p} \sin^2(\theta_i) = \delta^2$. Without loss of generality, the codebook may be rotated so that $W_i = (I\,0)^T$ and $W_j = (\mathrm{diag}(\cos\theta)\,\mathrm{diag}(\sin\theta))^T$ [15].

The midpoint between $[W_i]$ and $[W_j]$ measured with chordal distance is the midpoint on the geodesic. The principal angles between the midpoint on the geodesic joining $[W_i]$ and $[W_j]$, and the codeword $[W_i]$ are thus $(\frac{\theta_1}{2}, \ldots, \frac{\theta_p}{2})$ [14]. It follows that the squared chordal distance between the midpoint on the geodesic and an extremity of the geodesic is

$$\varrho^2 = \| \sin \frac{\vec{\theta}}{2} \|_2^2 = \sum_{i=1}^{p} \sin^2 \frac{\theta_i}{2}.$$

Finding lower and upper bounds reduces to solving the following optimization problems:

$$\begin{array}{ll} \underset{\vec{\theta} \in [0,\ \frac{\pi}{2}]^p}{\text{minimize/maximize}} & \| \sin \frac{\vec{\theta}}{2} \|_2^2 \\ \text{subject to} & \| \sin \vec{\theta} \|_2^2 = \delta^2. \end{array} \tag{29}$$

The corresponding Lagrange function is

$$\Lambda(\theta_1, \ldots, \theta_p, \lambda) = \| \sin \frac{\vec{\theta}}{2} \|_2^2 + \lambda \Big(\| \sin \vec{\theta} \|_2^2 - \delta^2 \Big).$$

Solving the following set of equations:

$$\begin{aligned} \frac{\partial \Lambda}{\partial \theta_i} &= \sin\theta_i (1/2 + 2\lambda \cos\theta_i) = 0 \quad \text{for } i = 1 \ldots p \\ \frac{\partial \Lambda}{\partial \lambda} &= \sum_{i=1}^{p} \sin^2 \theta_i - \delta^2 = 0 \end{aligned}$$

yields a set of stationary points where at least x angles are nonzero such that $x \geq \lceil \delta^2 \rceil$ and equal to $\theta^* = \arcsin \frac{\delta}{\sqrt{x}}$. It is then easy to verify that the objective funtion $f(x) = \sum_{i=1}^{p} \sin^2 \frac{\theta_i}{2} = x/2(1 - \sqrt{1 - \delta^2/x})$ is a strictly decreasing function on $[\lceil \delta^2 \rceil,\, p]$ and thus is minimized for $x = p$. The result follows.

Maximization in (29) is obtained when a minimum number of angles is maximized, i.e., with $(\theta_1^\star, \ldots, \theta_p^\star) \in [0,\ \frac{\pi}{2}]^p$ such that $\theta_1^\star = \cdots = \theta_{\lfloor \delta^2 \rfloor}^\star = \frac{\pi}{2}$, $\theta_{\lceil \delta^2 \rceil}^\star = \arcsin(\sqrt{\delta^2 - \lfloor \delta^2 \rfloor})$ and $\theta_{\lceil \delta^2 \rceil + 1}^\star = \cdots = \theta_p^\star = 0$.

This can be verify by contradiction: Defining $s_i = \sin^2 \theta_i$ and $t(s_i) = (1 - \sqrt{1 - s_i})/2$, consider the equivalent problem of maximizing $\sum t(s_i)$ such that $\sum s_i = \delta^2$ and without loss of generality $1 \geq s_1 \geq s_2 \cdots \geq s_p \geq 0$. By contradiction, assume that $\sum t(si)$ is maximum at $\vec{a}$ with $a_i > 0 \,\forall i$. It is possible to find a $\vec{b}$ with $b_i \geq a_i$ for $1 \leq i \leq p-1$ and $b_p = 0$. From the mean value theorem and since $t'(\cdot)$ is stricly increasing, there exist $c \in]a_{p-1},\, b_{p-1}[$ and $d \in]0,\, a_p[$ such that

$$\begin{aligned} \sum t(b_i) - t(a_i) &\geq t'(c) \sum_{i=1}^{p-1} (b_i - a_i) + t'(d)(a_p - b_p) \\ &= (t'(c) + t'(d)) a_p > 0, \end{aligned}$$

where the last equality is due to the constraint $\sum b_i = \sum a_i = \delta^2$. This is in contradiction with the fact that $\sum t(a_i)$ is a maximum. Repeating the procedure from s_p to $s_{\lceil \delta^2 \rceil}$ leads to the conclusion. Then, the maximum is

$$\begin{aligned} \sum_{i=1}^{p} \sin^2 \frac{\theta_i^\star}{2} &= \frac{\lfloor \delta^2 \rfloor}{2} + \frac{1 - \sqrt{1 - (\delta^2 - \lfloor \delta^2 \rfloor)}}{2} \\ &= \frac{1}{2} \left(\lceil \delta^2 \rceil - \sqrt{\lceil \delta^2 \rceil - \delta^2} \right). \end{aligned}$$

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