

# Uplink-Downlink Duality in Normalized MSE or SINR under Imperfect Channel Knowledge

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**Abstract**—Duality between the multi-antenna multi-user uplink and the downlink has been discovered in terms of sum rate, capacity region, signal-to-interference-plus-noise-ratio (SINR) region or normalized mean-squared error (MSE) region. Previous work on duality has assumed perfect channel knowledge. However, channel estimation is never perfect in practice. In this paper, channel estimation error as well as antenna correlation at the base station (BS) is taken into account. A multi-user system with multiple antennas at the BS and with single-antenna users is studied. Joint detection and transmission are used in the uplink and the downlink, respectively. It is analytically shown that with imperfect channel state information (CSI), under the same sum power constraint, the achievable SINR regions or the normalized MSE regions in both links are the same, as in the case with perfect CSI. Monte Carlo simulation results and discussions are also provided to complement the analysis.

## I. INTRODUCTION

During the past decade, multi-antenna systems have been subject to extensive research. Recently, the focus has been shifting from single-user to multi-user multi-antenna systems.

In multi-antenna multi-user systems, from the information-theoretic point of view, the multiple-access channel (MAC, uplink) is better understood than the broadcast channel (BC, downlink), due to their differences in interference and cooperation [1][2]. From the viewpoint of signal processing, the uplink is also easier to deal with than the downlink [3]-[6]. However, the uplink and the dual downlink are inherently related. In [7], the capacity region of scalar Gaussian MAC is shown to be equal to the capacity region of the dual scalar Gaussian BC with the same noise power at each receiver and under the same sum power constraint. In [8], a duality is established between the dirty-paper achievable rate region of Gaussian multiple-input multiple-output (MIMO) BC and the capacity region of Gaussian MIMO MAC. Furthermore, in a system with multiple antennas at the BS and with single-antenna users, under perfect channel knowledge, with the same sum power, the achievable SINR regions and normalized MSE regions for both links are the same, when noise variances are identical at all receivers [4][9][10]. Because of duality, problems in the downlink can be solved by forming and solving a dual uplink problem [4]-[6], which is very convenient.

Most previous research on duality has assumed perfect CSI. In practice, CSI is obtained from channel estimation and is always imperfect. We therefore need to account for this in

system design. In [11], a multi-user system with multiple antennas at the BS and with single-antenna users is studied. There it is shown that with zero-forcing (ZF) joint detection in the uplink and ZF joint transmission in the dual downlink, the bit error probabilities (BEPs) in both links are the same under perfect CSI. When there is channel estimation error, the BEPs in both links are not exactly the same, but the effects of channel estimation error on the BEPs of both links are comparable [11]. This motivates us to find out whether duality exists in term of other performance metrics between the uplink and the dual downlink under imperfect CSI. Although the result in [11] is enlightening, the analysis there is based on a linear Taylor series approximation of the channel estimation error and cannot be easily generalized. A more general analytical framework is desired, which can be used for performance metrics other than BEP or for non-ZF beamforming.

Since many performance criteria in communication system designs are related to MSE or SINR [12], in this paper, we use the framework in [9][10] and show by analysis that the uplink-downlink duality holds in normalized MSE and SINR regions with imperfect channel estimation. Our result extends the duality analysis in [9][10] to the case of imperfect channel estimation and antenna correlation at the base station. The approach here is different from that in [11]. Simulation results are provided, which illustrate our analysis and agree with previous analysis and simulation results in [11].

Notation: upper (lower) case boldface letters are for matrices (vectors);  $E\{\cdot\}$  denotes statistical expectation and  $tr(\cdot)$  stands for trace;  $|a|$  denotes the magnitude of complex scalar  $a$ ;  $(\cdot)^*$  means complex conjugate and  $(\cdot)^H$  means complex conjugate transpose (Hermitian);  $\mathcal{N}_c(\cdot, \cdot)$  denotes the complex Gaussian distribution;  $\mathbf{I}$  is reserved for the identity matrix and  $\mathbf{1}$  for the all-one column vector;  $diag\{\dots\}$  is a diagonal matrix containing entries within the brackets.

## II. SYSTEM MODEL AND PROBLEM STATEMENT

### A. System model

Consider a single cell in a cellular system with  $K$  users, each with a single antenna. The BS has  $M$  antennas. Assume  $M \geq K$ . The uplink channel is described by a  $M \times K$  matrix  $\mathbf{H} = [\mathbf{h}_1, \dots, \mathbf{h}_K]$ , and the column vector  $\mathbf{h}_i$  denotes the channel from user  $i$  to the BS,  $i = 1, \dots, K$ . We consider

correlation among the BS antennas. The uplink channel model with correlation is described by [13]:

$$\mathbf{H} = \mathbf{R}^{1/2} \mathbf{H}_w, \quad (1)$$

where  $\mathbf{R}$  is the normalized correlation matrix at the BS with unit diagonal entries.  $\mathbf{H}_w$  denotes a spatially white matrix, whose entries are independent and identically distributed (i.i.d.) with a complex Gaussian distribution  $\mathcal{N}_c(0, 1)$ . The corresponding dual downlink channel is simply denoted by  $\mathbf{H}^H$ .

The uplink is described by Fig. 1 [9][10]. Here signals from each user are i.i.d. with unit-energy and are individually power-controlled by a diagonal matrix  $\mathbf{Q}^{1/2}$ , where  $\mathbf{Q} = \text{diag}\{q_1, \dots, q_K\}$ . The signals are then transmitted through a Rayleigh flat-fading channel. The noise vector  $\mathbf{w}$  has i.i.d. entries  $\mathcal{N}_c(0, \sigma_n^2)$  and is independent of the data or channel realizations. At the receiver (BS), a joint detector (denoted by  $\mathbf{G}$ ) is used to separate different users, which corresponds to a normalized beamforming set  $\mathbf{U}^H$  followed by a diagonal matrix  $\mathbf{\Theta}$  in Fig. 1. Thus  $\mathbf{G} = \mathbf{\Theta} \mathbf{U}^H$ ,  $\mathbf{\Theta} = \text{diag}\{\theta_1, \dots, \theta_K\}$ , and  $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_K]$  with  $\|\mathbf{u}_j\|^2 = 1, j = 1, \dots, K$ . Here the columns of  $\mathbf{G}^H$  are normalized and then collected in  $\mathbf{U}$ , whereas the column norms are the diagonal entries of  $\mathbf{\Theta}$ . The signal received from the  $k$ -th user ( $k = 1, \dots, K$ ) can be written as

$$\hat{x}_k^{UL} = \frac{\theta_k}{\sqrt{q_k}} \mathbf{u}_k^H \left\{ \left[ \sum_{j=1}^K \mathbf{h}_j \cdot x_j^{UL} \cdot \sqrt{q_j} \right] + \mathbf{w} \right\} \quad (2)$$

$$= \frac{\theta_k}{\sqrt{q_k}} \mathbf{u}_k^H \left[ \mathbf{H} \mathbf{Q}^{1/2} \mathbf{x}^{UL} + \mathbf{w} \right]. \quad (3)$$

The downlink signal model is given in Fig. 2 [9][10]. The signal vector  $\mathbf{x}^{DL}$  with i.i.d. unit-energy entries denotes the symbols intended for each user. The signal vector is then power-controlled by the diagonal matrix  $\mathbf{P}^{1/2}$ , where  $\mathbf{P} = \text{diag}\{p_1, \dots, p_K\}$ . The matrix  $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_K]$  is the collection of normalized transmit signatures (beamformers) for different users, i.e.,  $\|\mathbf{u}_j\|^2 = 1, j = 1, \dots, K$ . The diagonal matrix  $\mathbf{\Theta} = \text{diag}\{\theta_1, \dots, \theta_K\}$  provides additional freedom to control the MSE performance [10]. The received signal of user  $k$  ( $k = 1, \dots, K$ ) is

$$\hat{x}_k^{DL} = \frac{\theta_k}{\sqrt{p_k}} \left\{ \mathbf{h}_k^H \left[ \sum_{j=1}^K \mathbf{u}_j \cdot x_j^{DL} \cdot \sqrt{p_j} \right] + n_k \right\} \quad (4)$$

$$= \frac{\theta_k}{\sqrt{p_k}} \left\{ \mathbf{h}_k^H \mathbf{U} \mathbf{P}^{1/2} \mathbf{x}^{DL} + n_k \right\}. \quad (5)$$

The noise of each user,  $n_k$  ( $k = 1, \dots, K$ ), is assumed to be i.i.d.  $\mathcal{N}_c(0, \sigma_n^2)$  and independent of the data vector.

Note that we have used the same notation for the uplink and downlink beamformers  $\{\mathbf{u}_j\}$  and scaling  $\{\theta_j\}$ ,  $j = 1, \dots, K$ . The purpose of this will become clear in the problem statement.

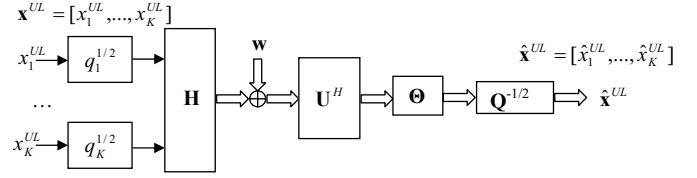


Fig. 1. Uplink signal model

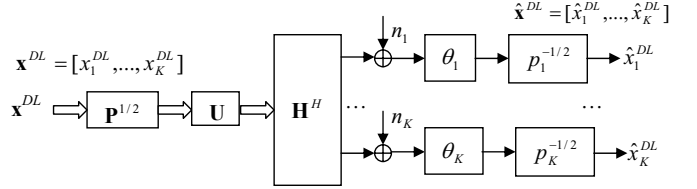


Fig. 2. Downlink signal model

### B. Channel estimation and the CSI model

The following CSI model is used for the uplink [14]:

$$\mathbf{H} = \hat{\mathbf{H}} + \mathbf{E} = \hat{\mathbf{H}} + \mathbf{R}_e^{1/2} \mathbf{E}_w. \quad (6)$$

In a vector form, (6) becomes

$$\mathbf{h}_k = \hat{\mathbf{h}}_k + \mathbf{e}_k = \hat{\mathbf{h}}_k + \mathbf{R}_e^{1/2} \mathbf{e}_{w,k}, \quad k = 1, \dots, K. \quad (7)$$

Here  $\mathbf{H}$  is the true channel,  $\hat{\mathbf{H}}$  is the linear minimum mean-squared error (LMMSE) channel estimate [16].  $\mathbf{R}_e$  is the equivalent correlation matrix resulting from channel estimation and  $\mathbf{R}_e = \mathbf{R}(\mathbf{R} + \sigma_{ce}^2 \mathbf{I})^{-1}$ , where  $\sigma_{ce}^2$  is the variance of channel estimation error and  $\mathbf{R}$  has been introduced earlier as the BS antenna correlation matrix. The matrix  $\mathbf{E}_w$  is spatially white, whose entries are i.i.d.  $\mathcal{N}_c(0, \sigma_{ce}^2)$ . Note that the entries of matrix  $\mathbf{E}$  ( $= \mathbf{R}_e^{1/2} \mathbf{E}_w$ ) are correlated. In the following, we assume that  $\hat{\mathbf{H}}$ ,  $\mathbf{R}$ ,  $\sigma_{ce}^2$  and  $\sigma_n^2$  are known at the BS, which are referred to as the CSI. The corresponding CSI model for the dual downlink is the Hermitian operation on (6) (or, equivalently, (7)).

### C. Problem statement

The objective is to determine whether the uplink and the dual downlink can achieve the same set of SINRs or normalized MSEs with the same sum power constraint, using the same set of normalized beamformers and scaling, and under the CSI model given above in Section II-B.

## III. DUALITY ANALYSIS UNDER IMPERFECT CSI

### A. Downlink

Consider the downlink channel. From (4), (5) and (7), when there is channel estimation error, the received signal at user  $k$  ( $k = 1, \dots, K$ ) can be expressed as

$$\hat{x}_k^{DL} = \theta_k \cdot \hat{\mathbf{h}}_k^H \mathbf{u}_k \cdot x_k^{DL} + I_{k,DL}, \quad (8)$$

where  $I_{k,DL}$  denotes the total interferences plus noise and

$$I_{k,DL} = \frac{\theta_k}{\sqrt{p_k}} \left[ \left( \sum_{j=1, j \neq k}^K \hat{\mathbf{h}}_k^H \mathbf{u}_j x_j^{DL} p_j^{1/2} \right) + \mathbf{e}_k^H \mathbf{U} \mathbf{P}^{1/2} \mathbf{x}^{DL} + n_k \right].$$

Here  $I_{k,DL}$  includes an interference term caused by channel estimation error.

The variance of  $\mathbf{e}_k^H \mathbf{U} \mathbf{P}^{1/2} \mathbf{x}^{DL}$  is calculated as [15]

$$\begin{aligned} & E\{\mathbf{e}_k^H \mathbf{U} \mathbf{P}^{1/2} \mathbf{x}^{DL} (\mathbf{x}^{DL})^H \mathbf{P}^{1/2} \mathbf{U}^H \mathbf{e}_k\} \\ &= E\{\mathbf{e}_{w,k}^H \mathbf{R}_e^{1/2} \mathbf{U} \mathbf{P} \mathbf{U}^H \mathbf{R}_e^{1/2} \mathbf{e}_{w,k}\} \\ &= \sigma_{ce}^2 \cdot \text{tr}\{\mathbf{R}_e \mathbf{U} \mathbf{P} \mathbf{U}^H\}. \end{aligned}$$

Using the uncorrelatedness among data symbols, channel noise and channel estimation error, we can show that the variance of  $I_{k,DL}$  ( $k = 1, \dots, K$ ) is

$$\begin{aligned} & E\{|I_{k,DL}|^2\} \\ &= \frac{\theta_k^2}{p_k} \left[ \sigma_n^2 + \sigma_{ce}^2 \cdot \text{tr}\{\mathbf{R}_e \mathbf{U} \mathbf{P} \mathbf{U}^H\} + \sum_{j=1, j \neq k}^K |\hat{\mathbf{h}}_k^H \mathbf{u}_j|^2 \cdot p_j \right]. \end{aligned} \quad (9)$$

It is clear that the SINR of user  $k$  is

$$\begin{aligned} & SINR_k^{DL} \\ &= \frac{|\hat{\mathbf{h}}_k^H \mathbf{u}_k|^2 \cdot p_k}{\sigma_n^2 + \sigma_{ce}^2 \cdot \text{tr}\{\mathbf{R}_e \mathbf{U} \mathbf{P} \mathbf{U}^H\} + \sum_{j=1, j \neq k}^K |\hat{\mathbf{h}}_k^H \mathbf{u}_j|^2 \cdot p_j}. \end{aligned} \quad (10)$$

The normalized MSE between  $\hat{x}_k^{DL}$  and  $x_k^{DL}$  is given by

$$\begin{aligned} \varepsilon_k^{DL} &= E\{|\hat{x}_k^{DL} - x_k^{DL}|^2\} \\ &= \frac{\theta_k^2}{p_k} \left[ \sigma_n^2 + \sigma_{ce}^2 \cdot \text{tr}\{\mathbf{R}_e \mathbf{U} \mathbf{P} \mathbf{U}^H\} + \sum_{j=1}^K |\hat{\mathbf{h}}_k^H \mathbf{u}_j|^2 \cdot p_j \right] \\ &\quad - \theta_k \cdot \hat{\mathbf{h}}_k^H \mathbf{u}_k - \theta_k \cdot \mathbf{u}_k^H \hat{\mathbf{h}}_k + 1. \end{aligned} \quad (11)$$

Define

$$\begin{aligned} a_k^{DL} &= \frac{SINR_k^{DL}}{(1 + SINR_k^{DL}) |\hat{\mathbf{h}}_k^H \mathbf{u}_k|^2}, \quad k = 1, \dots, K, \\ &= \frac{p_k}{\sigma_n^2 + \sigma_{ce}^2 \cdot \text{tr}(\mathbf{R}_e \mathbf{U} \mathbf{P} \mathbf{U}^H) + \sum_{j=1}^K |\hat{\mathbf{h}}_k^H \mathbf{u}_j|^2 \cdot p_j}. \end{aligned}$$

Then we have

$$p_k = \left[ \sigma_n^2 + \sigma_{ce}^2 \text{tr}(\mathbf{R}_e \mathbf{U} \mathbf{P} \mathbf{U}^H) + \sum_{j=1}^K |\hat{\mathbf{h}}_k^H \mathbf{u}_j|^2 \cdot p_j \right] a_k^{DL}. \quad (12)$$

Let  $\mathbf{p} = [p_1, \dots, p_K]^T$  and  $\mathbf{D}_{DL} = \text{diag}\{a_1^{DL}, \dots, a_K^{DL}\}$ . Define a matrix  $\Phi$  whose  $(i, j)$ -th element is given by  $|\mathbf{u}_j^H \hat{\mathbf{h}}_i|^2$ ,  $1 \leq i, j \leq K$ . Based on (12),

$$\mathbf{p} = [\sigma_n^2 + \sigma_{ce}^2 \cdot \text{tr}(\mathbf{R}_e \mathbf{U} \mathbf{P} \mathbf{U}^H)] \cdot \mathbf{D}_{DL} \cdot \mathbf{1} + \mathbf{D}_{DL} \cdot \Phi \cdot \mathbf{p},$$

which is equivalent to

$$\mathbf{p} = [\sigma_n^2 + \sigma_{ce}^2 \cdot \text{tr}(\mathbf{R}_e \mathbf{U} \mathbf{P} \mathbf{U}^H)] \cdot (\mathbf{D}_{DL}^{-1} - \Phi)^{-1} \mathbf{1}. \quad (13)$$

## B. Uplink

Now we turn to the uplink channel. Under imperfect CSI, the received signal of user  $k$  ( $k = 1, \dots, K$ ) at the BS is expressed as

$$\hat{x}_k^{UL} = \theta_k \cdot \mathbf{u}_k^H \hat{\mathbf{h}}_k \cdot x_k^{UL} + I_{k,UL}, \quad (14)$$

where  $I_{k,UL}$  is the total noise and interference and

$$I_{k,UL} = \frac{\theta_k}{\sqrt{q_k}} \left[ \left( \sum_{j=1, j \neq k}^K \mathbf{u}_k^H \hat{\mathbf{h}}_j x_j^{UL} q_j^{1/2} \right) + \mathbf{u}_k^H \mathbf{E} \mathbf{Q}^{1/2} \mathbf{x}^{UL} + \mathbf{u}_k^H \mathbf{w} \right].$$

The variance of  $\mathbf{u}_k^H \mathbf{E} \mathbf{Q}^{1/2} \mathbf{x}^{UL}$  is

$$\begin{aligned} & E\{\mathbf{u}_k^H \mathbf{E} \mathbf{Q}^{1/2} \mathbf{x}^{UL} (\mathbf{x}^{UL})^H \mathbf{Q}^{1/2} \mathbf{E}^H \mathbf{u}_k\} \\ &= \mathbf{u}_k^H \mathbf{R}_e^{1/2} E\{\mathbf{E} \mathbf{w} \mathbf{Q} \mathbf{E}^H\} \mathbf{R}_e^{1/2} \mathbf{u}_k \\ &= \mathbf{u}_k^H \mathbf{R}_e^{1/2} \{\sigma_{ce}^2 \cdot \text{tr}(\mathbf{Q}) \cdot \mathbf{I}\} \mathbf{R}_e^{1/2} \mathbf{u}_k \\ &= \sigma_{ce}^2 \cdot \text{tr}(\mathbf{Q}) \cdot \mathbf{u}_k^H \mathbf{R}_e \mathbf{u}_k, \end{aligned}$$

and then the variance of  $I_{k,UL}$  ( $k = 1, \dots, K$ ) is given by

$$\begin{aligned} & E\{|I_{k,UL}|^2\} \\ &= \frac{\theta_k^2}{q_k} \left[ \sigma_n^2 + \sigma_{ce}^2 \text{tr}\{\mathbf{Q}\} \cdot \mathbf{u}_k^H \mathbf{R}_e \mathbf{u}_k + \sum_{j=1, j \neq k}^K |\mathbf{u}_k^H \hat{\mathbf{h}}_j|^2 q_j \right]. \end{aligned} \quad (15)$$

In the above, we have used the fact that  $\|\mathbf{u}_k\|^2 = 1$ ,  $k = 1, \dots, K$ , and the uncorrelatedness among data symbols, channel noise and channel estimation error. Thus the SINR of user  $k$  ( $k = 1, \dots, K$ ) is equal to

$$\begin{aligned} & SINR_k^{UL} \\ &= \frac{q_k \cdot |\mathbf{u}_k^H \hat{\mathbf{h}}_k|^2}{\sigma_n^2 + \sigma_{ce}^2 \text{tr}\{\mathbf{Q}\} \cdot \mathbf{u}_k^H \mathbf{R}_e \mathbf{u}_k + \sum_{j=1, j \neq k}^K |\mathbf{u}_k^H \hat{\mathbf{h}}_j|^2 q_j}. \end{aligned} \quad (16)$$

Correspondingly, the normalized MSE of user  $k$  is

$$\begin{aligned} \varepsilon_k^{UL} &= E\{|\hat{x}_k^{UL} - x_k^{UL}|^2\} \\ &= \frac{\theta_k^2}{q_k} \left[ \sigma_n^2 + \sigma_{ce}^2 \text{tr}\{\mathbf{Q}\} \cdot \mathbf{u}_k^H \mathbf{R}_e \mathbf{u}_k + \sum_{j=1}^K |\mathbf{u}_k^H \hat{\mathbf{h}}_j|^2 q_j \right] \\ &\quad - \theta_k \cdot \mathbf{u}_k^H \hat{\mathbf{h}}_k - \theta_k \cdot \hat{\mathbf{h}}_k^H \mathbf{u}_k + 1. \end{aligned} \quad (17)$$

Let

$$\begin{aligned} a_k^{UL} &= \frac{SINR_k^{UL}}{(1 + SINR_k^{UL}) |\mathbf{u}_k^H \hat{\mathbf{h}}_k|^2}, \quad k = 1, \dots, K, \\ &= \frac{q_k}{\sigma_n^2 + \sigma_{ce}^2 \cdot \text{tr}(\mathbf{Q}) \cdot \mathbf{u}_k^H \mathbf{R}_e \mathbf{u}_k + \sum_{j=1}^K |\mathbf{u}_k^H \hat{\mathbf{h}}_j|^2 \cdot q_j}. \end{aligned}$$

Then

$$q_k = \left[ \sigma_n^2 + \sigma_{ce}^2 \text{tr}(\mathbf{Q}) \cdot \mathbf{u}_k^H \mathbf{R}_e \mathbf{u}_k + \sum_{j=1}^K |\mathbf{u}_k^H \hat{\mathbf{h}}_j|^2 \cdot q_j \right] a_k^{UL}. \quad (18)$$

Let  $\mathbf{q} = [q_1, \dots, q_K]^T$ ,  $\mathbf{D}_{UL} = \text{diag}\{a_1^{UL}, \dots, a_K^{UL}\}$ , and  $\mathbf{b} = [\mathbf{u}_1^H \mathbf{R}_e \mathbf{u}_1, \dots, \mathbf{u}_K^H \mathbf{R}_e \mathbf{u}_K]^T$ . From (18),

$$\mathbf{q} = \sigma_n^2 \cdot \mathbf{D}_{UL} \cdot \mathbf{1} + \sigma_{ce}^2 \cdot \text{tr}(\mathbf{Q}) \cdot \mathbf{D}_{UL} \cdot \mathbf{b} + \mathbf{D}_{UL} \cdot \Phi^T \cdot \mathbf{q}.$$

Therefore,

$$\mathbf{q} = \sigma_n^2 (\mathbf{D}_{UL}^{-1} - \Phi^T)^{-1} \mathbf{1} + \sigma_{ce}^2 \text{tr}(\mathbf{Q}) (\mathbf{D}_{UL}^{-1} - \Phi^T)^{-1} \mathbf{b}. \quad (19)$$

### C. Results

**Result 1: (Relation between SINR and normalized MSE under imperfect CSI)** Assume the same sets of  $\{\mathbf{u}_j\}$  and  $\{\theta_j\}$ ,  $j = 1, \dots, K$  are used for both the uplink and the downlink. Assume  $\mathbf{p}$  ( $\mathbf{q}$ ) achieves the downlink (uplink)  $\text{SINR}_k^{DL}$  ( $\text{SINR}_k^{UL}$ ) for user  $k$ ,  $k = 1, \dots, K$ . If  $\text{SINR}_k^{DL}$  is equal to  $\text{SINR}_k^{UL}$ , then the normalized MSE  $\varepsilon_k^{DL} = \varepsilon_k^{UL}$ ,  $k = 1, \dots, K$ , and vice versa.

**Proof:** Using (10), (11), (16) and (17), the result is straightforward.  $\square$

**Result 2: (Duality in normalized MSE or SINR with channel estimation error and antenna correlation at BS)** If the uplink and downlink employ the same sets of  $\{\mathbf{u}_j\}$  and  $\{\theta_j\}$ ,  $j = 1, \dots, K$ , and they achieve the same set of SINRs (or normalized MSEs) under imperfect CSI, then the sum powers in both links are equal.

**Proof:** Consider the downlink case. Let  $\alpha$  be equal to  $[\sigma_n^2 + \sigma_{ce}^2 \cdot \text{tr}(\mathbf{R}_e \mathbf{U} \mathbf{P} \mathbf{U}^H)]$  in (13), i.e.,

$$\mathbf{p} = \alpha \cdot (\mathbf{D}_{DL}^{-1} - \Phi)^{-1} \mathbf{1}. \quad (20)$$

Note that

$$\text{tr}(\mathbf{P}) = \mathbf{1}^T \mathbf{p} = \alpha \cdot \mathbf{1}^T (\mathbf{D}_{DL}^{-1} - \Phi)^{-1} \mathbf{1}, \quad (21)$$

$$\text{tr}(\mathbf{R}_e \mathbf{U} \mathbf{P} \mathbf{U}^H) = \sum_{j=1}^K p_j \cdot \mathbf{u}_j^H \mathbf{R}_e \mathbf{u}_j = \mathbf{p}^T \mathbf{b}, \quad (22)$$

$$\mathbf{1}^T (\mathbf{D}_{DL}^{-1} - \Phi)^{-1} \mathbf{1} = \mathbf{1}^T (\mathbf{D}_{DL}^{-1} - \Phi^T)^{-1} \mathbf{1}. \quad (23)$$

Then we obtain

$$\begin{aligned} \mathbf{1}^T \mathbf{p} &\stackrel{(13)(22)}{=} (\sigma_n^2 + \sigma_{ce}^2 \cdot \mathbf{p}^T \mathbf{b}) \cdot \mathbf{1}^T (\mathbf{D}_{DL}^{-1} - \Phi)^{-1} \mathbf{1}, \\ &\stackrel{(21)}{=} \sigma_n^2 \cdot \mathbf{1}^T (\mathbf{D}_{DL}^{-1} - \Phi)^{-1} \mathbf{1} + \sigma_{ce}^2 \cdot \mathbf{p}^T \mathbf{b} \cdot \text{tr}(\mathbf{P}) / \alpha, \\ &\stackrel{(20)(23)}{=} \sigma_n^2 \cdot \mathbf{1}^T (\mathbf{D}_{DL}^{-1} - \Phi^T)^{-1} \mathbf{1} \\ &\quad + \sigma_{ce}^2 \cdot \text{tr}(\mathbf{P}) \cdot \mathbf{1}^T (\mathbf{D}_{DL}^{-1} - \Phi^T)^{-1} \mathbf{b}. \end{aligned}$$

Therefore,

$$\text{tr}(\mathbf{P}) = \frac{\sigma_n^2 \cdot \mathbf{1}^T (\mathbf{D}_{DL}^{-1} - \Phi^T)^{-1} \mathbf{1}}{1 - \sigma_{ce}^2 \cdot \mathbf{1}^T (\mathbf{D}_{DL}^{-1} - \Phi^T)^{-1} \mathbf{b}}.$$

For the uplink, from (19),

$$\begin{aligned} \mathbf{1}^T \mathbf{q} = \text{tr}(\mathbf{Q}) &= \sigma_n^2 \cdot \mathbf{1}^T (\mathbf{D}_{UL}^{-1} - \Phi^T)^{-1} \mathbf{1} \\ &\quad + \sigma_{ce}^2 \cdot \text{tr}(\mathbf{Q}) \cdot \mathbf{1}^T (\mathbf{D}_{UL}^{-1} - \Phi^T)^{-1} \mathbf{b}, \end{aligned}$$

and thus

$$\text{tr}(\mathbf{Q}) = \frac{\sigma_n^2 \cdot \mathbf{1}^T (\mathbf{D}_{UL}^{-1} - \Phi^T)^{-1} \mathbf{1}}{1 - \sigma_{ce}^2 \cdot \mathbf{1}^T (\mathbf{D}_{UL}^{-1} - \Phi^T)^{-1} \mathbf{b}}.$$

When the uplink and the downlink achieve the same set of SINRs (or normalized MSEs, based on **Result 1**), i.e.,  $\mathbf{D}_{UL} = \mathbf{D}_{DL}$ , we must have  $\text{tr}(\mathbf{Q}) = \text{tr}(\mathbf{P})$ . This concludes the proof of **Result 2**.  $\square$

Based on the above proof, **Result 2** can also be expressed as follows: If  $\text{tr}(\mathbf{P}) = \text{tr}(\mathbf{Q})$  and the same sets of  $\{\mathbf{u}_j\}$  and  $\{\theta_j\}$ ,  $j = 1, \dots, K$ , are used for both links, then the set of SINRs (or normalized MSEs) is achievable in the downlink if and only if it is achievable in the uplink, under the **imperfect CSI**.

The implications of the above two results are that when channel estimation is imperfect at the BS, SINR or normalized MSE based downlink system design problems can still be solved by dealing with a dual uplink problem.

## IV. NUMERICAL RESULTS

In this section, we provide numerical results to complement the analysis. In the following,  $K = M = 5$  and  $\sigma_n^2 = 1$ . Let  $P_T$  represent the sum power for all users, i.e.,  $P_T = \text{tr}(\mathbf{P}) = \text{tr}(\mathbf{Q})$ . The correlation model for the BS antennas is chosen as in [17]:  $(\mathbf{R})_{ij} = \rho^{|i-j|}$ ,  $0 \leq \rho < 1$ ,  $i, j \in \{1, \dots, n_T\}$ . 4-QAM is used for each user's data stream in both links.

For the uplink, we assume uniform power allocation among users, i.e.,  $\mathbf{q} = (P_T/K) \cdot \mathbf{1}$ . The LMMSE joint detection is employed at the BS based on the channel estimates. For the downlink, we use the duality results to find the joint transmission strategy. For each channel realization, we first find  $\mathbf{U}$ ,  $\Theta$  and  $\Phi$  from the dual uplink using LMMSE joint detection, given the channel estimate. We also calculate the SINRs for the uplink. The power allocation matrix  $\mathbf{P}$  is then calculated using (13). After obtaining  $\mathbf{U}$  and  $\mathbf{P}$ , joint transmission for the downlink can be performed. Without loss of generality, we calculate the average normalized MSE and BEP of user 1 from Monte Carlo simulations.

Fig. 3 shows the BEP results. When channel estimation is perfect, the BEPs of user 1 in both links are equal. If there is channel estimation error, the two BEP curves diverge when  $P_T$  is large and channel estimation error becomes the dominant source of errors. A similar phenomenon has been observed in [11], when zero-forcing beamformers are used for both links. Based on the analysis and simulations in [11] as well as Fig. 3, the BEPs of user 1 in both links are not the same at (relatively) high SNR under imperfect CSI. However, from Fig. 4, we can see that both links achieve the same average normalized MSEs, with or without channel estimation error. This agrees with our analysis in previous section<sup>1</sup>. Similar

<sup>1</sup>Note that in (10), (11), (16) and (17), we have used the statistical averages of the interference-plus-noise powers in the calculations of SINRs or normalized MSEs. In practice and in our Monte Carlo simulations, we use the actual realizations of the interference-plus-noise powers. Therefore, one should not be surprised if there is a discrepancy between the simulation results and the analysis. However, as shown by Fig. 4, the simulation results and the analysis agree quite well for the normalized MSE.

results can be observed for other values of  $\rho$  or with non-uniform power allocations in the uplink.

We now provide more insight into the results in Fig. 3 and Fig. 4. From (10), (11), (16) and (17), we can see that the derivations of SINRs or normalized MSEs depend largely on the second-order statistics of fading, channel noise and channel estimation error, which are the same for the dual links. Therefore, the normalized MSEs of both links seem to be the same. On the other hand, the BEPs depend on the interference-plus-noise distributions, which vary with the relative strengths of the interferences. When there is channel estimation error, the interference strengths depend on  $P_T$  and this dependence is possibly different for the uplink and downlink (See (9) and (15)). Therefore, the BEPs become noticeably unequal when  $P_T$  is relatively large (and thus channel estimation error becomes the dominant source of errors).

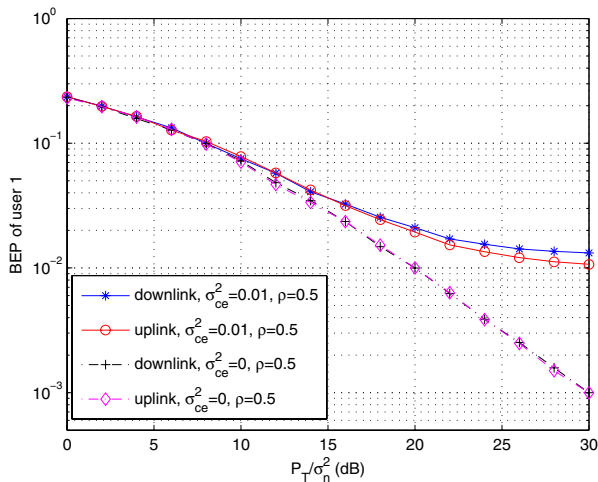


Fig. 3. BEP performance of user 1,  $K = M = 5$ .

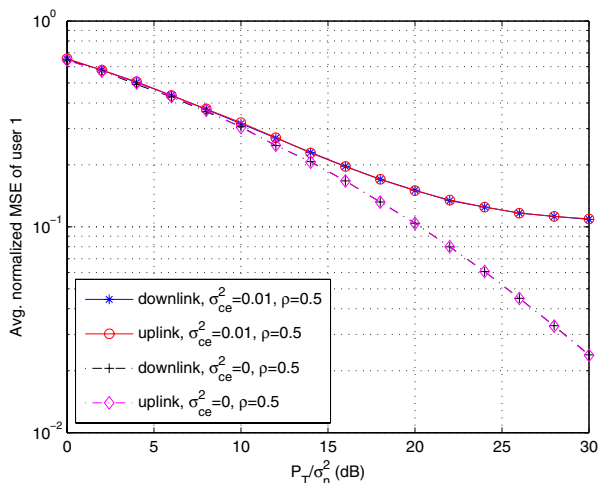


Fig. 4. Average normalized MSE performance of user 1,  $K = M = 5$ .

## V. CONCLUSIONS

We have investigated the uplink-downlink duality in a system with multiple antennas at the BS and with single-antenna users under imperfect CSI at the BS. We have shown by analysis that duality holds in terms of SINR or normalized MSE regions under imperfect CSI. Simulations results for the normalized MSEs with LMMSE joint detection in the uplink and LMMSE joint transmission in the downlink verify our analysis.

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